On the number of hyperbolic 3-manifolds of a given volume

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Abstract. The work of Jørgensen and Thurston shows that there is a finite number $N(v)$ of orientable hyperbolic 3-manifolds with any given volume v. We show that there is an infinite sequence of closed orientable hyperbolic 3-manifolds, obtained by Dehn filling on the figure eight knot complement, that are uniquely determined by their volumes. This gives a sequence of distinct volumes x_i converging to the volume of the figure eight knot complement with $N(x_i) = 1$ for each i. We also give an infinite sequence of 1-cusped hyperbolic 3-manifolds, obtained by Dehn filling one cusp of the $(-2, 3, 8)$-pretzel link complement, that are uniquely determined by their volumes amongst orientable cusped hyperbolic 3-manifolds. Finally, we describe examples showing that the number of hyperbolic link complements with a given volume v can grow at least exponentially fast with v.

1. Introduction

Thurston and Jørgensen (see [**43**]) showed that the set of volumes of finite volume orientable complete hyperbolic 3-manifolds is a closed, non-discrete, well ordered subset of $\mathbb{R}_{>0}$ of order type ω^ω. Further, the number $N(v)$ of complete orientable hyperbolic 3-manifolds of volume v is finite for all $v \in \mathbb{R}_{>0}$. In this paper we study how $N(v)$ varies with v.

There has been much work on determining the lowest volumes in various classes of hyperbolic 3-manifolds. The work of Gabai-Meyerhoff-Milley ([**21, 22, 35**]) shows that the Weeks manifold with volume $v_1 = 0.9427\ldots$ is the unique orientable hyperbolic 3-manifold of lowest volume, so that $N(v_1) = 1$. Prior to our work, this was the only known exact value for $N(v)$.

It is also interesting to look at the number of hyperbolic 3-manifolds of a given volume in various special classes. For example, we could study the numbers $N_C(v)$, $N_A(v)$, $N_L(v)$, $N_G(v)$ of orientable hyperbolic 3-manifolds of volume v that are cusped, arithmetic, link complements, or with geodesic boundary, respectively.

The work of Cao-Meyerhoff [**9**] shows there are precisely two orientable cusped hyperbolic 3-manifolds of lowest volume $v_\omega \approx 2.029883\ldots$, the figure eight knot complement and its sister (obtained by (-5,1) surgery on the Whitehead link complement). (See Figure 1.)

Gabai-Meyerhoff-Milley [**21**] determined the first 10 low volume orientable cusped hyperbolic 3-manifolds. This shows that the next 4 limit volumes are:

$$v_{2\omega} = 2.568970\ldots, v_{3\omega} = 2.666744\ldots, v_{4\omega} = 2.781833\ldots, v_{5\omega} = 2.828122\ldots$$

and that $N_C(v_\omega) = 2, N_C(v_{2\omega}) = 2, N_C(v_{3\omega}) = 2, N_C(v_{4\omega}) = 1, N_C(v_{5\omega}) = 3.$

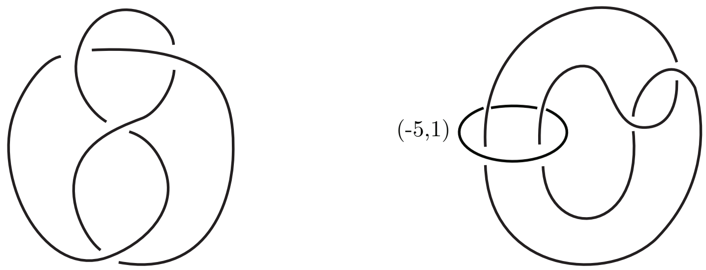

FIGURE 1. The figure eight knot and its sister, with complements denoted $m004$ and $m003$ respectively in SnapPea notation.

The work of Agol [**4**] shows that there are exactly two orientable 2-cusped hyperbolic 3-manifolds of lowest volume $v_{\omega^2} = 3.663862\ldots$, namely the Whitehead link complement and the $(-2,3,8)$-pretzel link complement. (See Figure 2.)

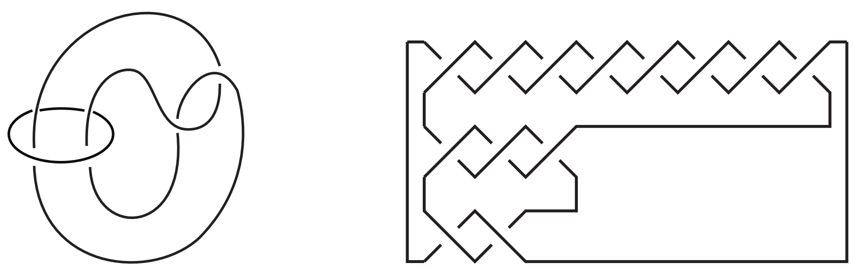

FIGURE 2. The Whitehead link and the $(-2,3,8)$-pretzel link, with complements $m129$ and $m125$ in SnapPea notation.

Chinburg-Friedman-Jones-Reid [**12**] showed that the Weeks manifold and the Meyerhoff manifold are the unique arithmetic hyperbolic 3-manifolds of lowest volume; hence $N_A(0.942707\ldots) = 1$ and $N_A(0.981368\ldots) = 1$.

Kojima and Miyamoto [**30, 36**] showed the lowest volume for a compact hyperbolic 3-manifold with geodesic boundary is $6.451990\ldots$, and Fujii [**19**] showed that there are exactly 8 such manifolds with this volume.

The graphs in Figures 3 and 4 below show the frequencies of volumes arising from the manifolds contained in the Callahan-Hildebrand-Weeks census of cusped hyperbolic 3-manifolds (see [**8**]), and the Hodgson-Weeks census of closed hyperbolic 3-manifolds (see [**29**]). These give *lower bounds* on $N(v)$, but note that infinitely many manifolds are definitely missing in these pictures. Nevertheless, the results suggest that there are often very few manifolds with any given volume.

The first main result of this paper (Theorem 3.1) shows that there is an infinite sequence of closed hyperbolic 3-manifolds determined by their volumes, obtained by Dehn fillings on the figure eight knot complement. Hence $N(x_i) = 1$ for an infinite sequence of volumes x_i converging to v_ω from below.

The proof of Theorem 3.1 uses the asymptotic formula of Neumann-Zagier [**38**] for the change in volume during hyperbolic Dehn filling and a study of the quadratic form giving the squares of lengths of closed geodesics on a horospherical torus cusp cross section, together with some elementary number theory.

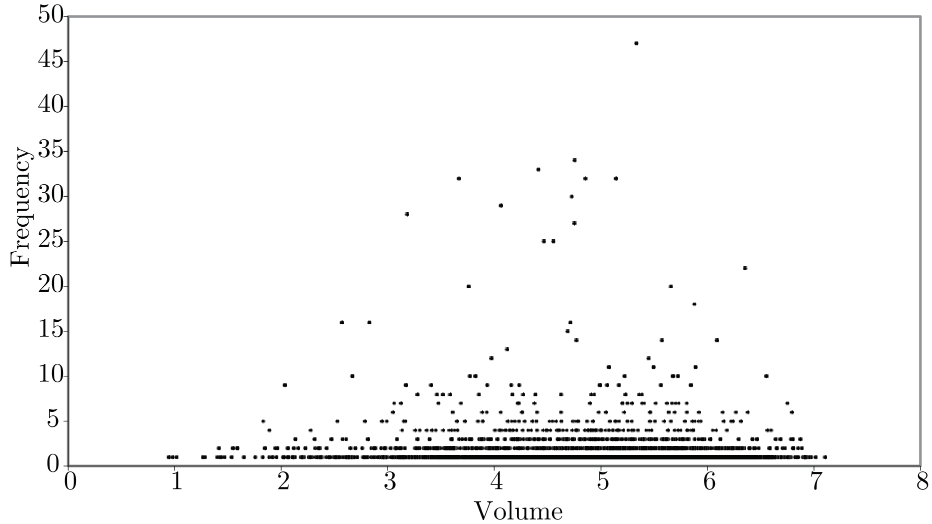

FIGURE 3. Frequencies of volumes of hyperbolic 3-manifolds in the closed and cusped censuses.

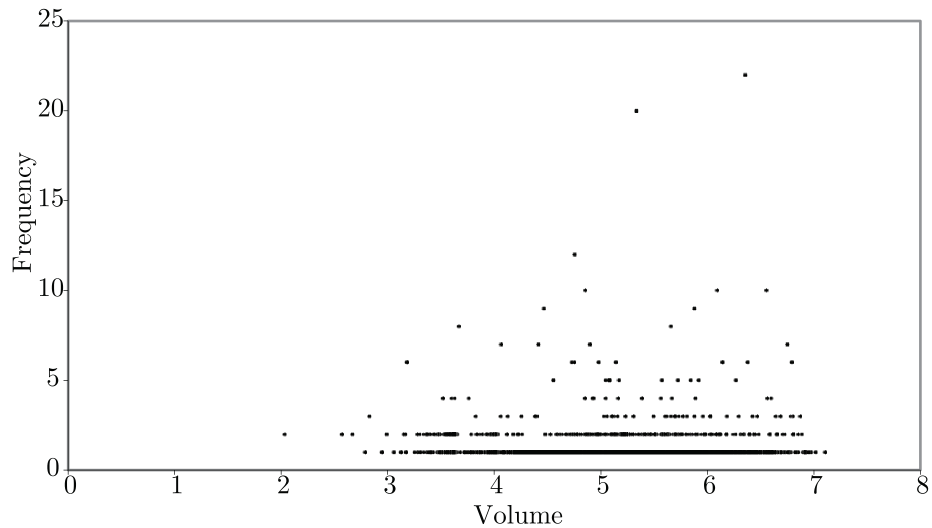

FIGURE 4. Frequencies of volumes of hyperbolic 3-manifolds in the cusped census.

Our second major result (Theorem 4.1) shows that there is an infinite sequence of 1-cusped hyperbolic 3-manifolds determined by their volumes amongst orientable cusped hyperbolic 3-manifolds. These manifolds are obtained by Dehn fillings on one cusp of the $(-2,3,8)$-pretzel link complement. Hence $N_C(y_i) = 1$ for an infinite sequence of volumes y_i converging to v_{ω^2} from below.

The proof of Theorem 4.1 is similar to the proof of Theorem 3.1, but there are some additional complications that require us to study higher order terms in the Neumann-Zagier asymptotic formula.

Another obvious feature of the graphs in Figures 3 and 4 is that there are special values of v for which $N(v)$ is particularly large. So it is natural to ask how fast $N(v)$ can grow with v.

If we fix a natural number n, then Wielenberg [**46**] showed that there exists v such that $N(v) > n$, and Zimmerman [**47**] constructed n distinct closed hyperbolic 3-manifolds which share the same volume. In both results, the number of manifolds of volume v constructed was at most linear in v.

More generally, by considering covering spaces of a fixed non-arithmetic hyperbolic 3-manifold whose fundamental group surjects onto a free group of rank 2, it is possible to find sequences of volumes $x_i \to \infty$ and a constant $c > 0$ such that $N(x_i) > x_i^{c\,x_i}$ for all i. This argument, known to Thurston and Lubotzky, can be found in [**10**, pp2633-2634] or [**6**, p1162].

Recent work of Belolipetsky-Gelander-Lubotzky-Shalev [**5**] gives very precise upper and lower bounds on the number $N_A(v)$ of *arithmetic* hyperbolic 3-manifolds of volume v: there exist constants $a, b > 0$ such that for $x >> 0$,

$$x^{ax} \leq \sum_{x_i \leq x} N_A(x_i) \leq x^{bx}.$$

In particular, this implies that for all sufficiently large x,

$$x^{ax} \leq \max_{x_i \leq x} N_A(x_i) \ \leq x^{bx}.$$

In [**18**] (and [**17**]), Frigerio, Martelli and Petronio constructed explicit families of hyperbolic 3-manifolds with totally geodesic boundary giving sequences $x_i \to \infty$ such that the number $N_G(x_i)$ of hyperbolic 3-manifolds of volume x_i with geodesic boundary grows at least as fast as $x_i^{cx_i}$ for some $c > 0$.

In the last part of this paper, we investigate the number $N_L(v)$ of hyperbolic link complements in S^3 with volume v. In Theorem 8.5 we construct sequences of volumes $v_n \to \infty$ such that the number $N_L(v_n)$ of hyperbolic link complements with volume v_n grows at least exponentially fast with v_n.

Our main tool is mutation along totally geodesic thrice punctured spheres in fully augmented link complements. It is well known that mutation along certain special surfaces leaves the hyperbolic volume and many other invariants unchanged (see for example [**2**], [**40**], [**15**]). We distinguish the manifolds produced using information about their cusp shapes and maximal horoball cusp neighbourhoods.

Recent independent work of Chesebro-DeBlois [**11**] uses mutations along 4-punctured spheres to show that $N_L(v)$ grows at least exponentially fast with v, and that the number of commensurability classes of link complements with a given volume can be arbitrarily large.

Mutation along thrice punctured spheres can be studied by using spatial trivalent graphs. We will say that a spatial trivalent graph is *hyperbolic* if its complement admits a hyperbolic metric with parabolic meridians and geodesic boundaries (see section 6, [**26**] and [**34**]). In [**34**], the second author exhibited volume preserving moves on hyperbolic graphs and hyperbolic links, which correspond to cutting and gluing along thrice punctured spheres. By applying these moves, each hyperbolic graph with a planar diagram corresponds to a *fully augmented link*. Such links have

been well studied, and we can easily compute the moduli of their cusps (see [**20**] and [**39**]).

Here is a brief outline of the remainder of the paper. In section 2, we recall the results of Neumann and Zagier relating the decrease in volume during hyperbolic Dehn filling to the length of the surgery curve. We then explain the key ideas that will allow us to obtain manifolds uniquely determined by their volumes. In section 3 (resp. 4), we apply these ideas to the manifolds denoted $m003$ and $m004$ (resp. $m125$ and $m129$) in SnapPea notation, and produce infinitely many manifolds that are determined by their volumes amongst all (resp. all cusped) orientable hyperbolic 3-manifolds. In section 5, we compute some higher order terms in the Neumann-Zagier asymptotic formula for the examples considered in sections 3 and 4; some of those computations are used in section 4. In section 6, we review some of the volume preserving moves on graphs from [**34**]. In section 7, we compute the moduli of the cusps of the manifolds that are obtained from the graph in Figure 8. Then in section 8, we construct manifolds corresponding to binary cyclic words of n letters and show that these manifolds are isometric if and only if their cyclic words are related by the natural action of the dihedral group D_n. Since the volume of these manifolds is a constant multiple of n, this gives a sequence $\mathcal{L}_n$ of families of distinct hyperbolic link complements which share the same volume v_n, such that the growth rate of $\#\mathcal{L}_n$ is at least exponential in v_n. Finally, in section 9, we collect some open problems related to the work in this paper.

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2. Hyperbolic Dehn fillings determined by their volumes

Let M be a cusped hyperbolic 3-manifold and let T be a horospherical torus cross section of one of the cusps of M. Choose an oriented basis for $H_1(T;\mathbb{Z}) \cong \mathbb{Z}^2$, called the "meridian" and "longitude" for T, and let $M(a,b)$ be the result of Dehn filling along the simple closed curve with homology class $(a,b) \in H_1(T;\mathbb{Z}) \cong \mathbb{Z}^2$.

Let A denote the area of T and $L(a,b)$ the length of the geodesic on T with homology class (a,b). Then we will study the quadratic form

$$Q(a,b) = \frac{L(a,b)^2}{A}. \tag{2.1}$$

(This gives the square of the length of (a,b) when the torus T is rescaled to have area 1, and is also known as the extremal length of the curve (a,b).)

Remark 2.1. *(Cusp Shape Parameters)* The horospherical torus T is similar to the Euclidean torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, with *shape parameter* $\tau \in \mathbb{C}$ defined by

$$\tau = (\text{complex translation of longitude})/(\text{complex translation of meridian}), \tag{2.2}$$

where the torus is viewed from the cusp. Note that with the standard orientation convention for a link complement in S^3 or for an oriented manifold in SnapPea,

this means that τ will have *negative* imaginary part.[1] Then we have

$$Q(a,b) = \frac{|a+b\tau|^2}{|\mathrm{Im}(\tau)|}. \tag{2.3}$$

The following result shows that the volume of $M(a,b)$ determines $Q(a,b)$ up to a uniformly bounded error. This observation will play a key role in our work.

Proposition 2.2. *Let V_0 be the volume of the complete hyperbolic structure on M. Then there exist constants $C_1 = C_1(M), C_2 = C_2(M) > 0$ such that*

(a) *the Dehn filling $M(a,b)$ is hyperbolic whenever $L(a,b) > C_1$, and*
(b) *for all hyperbolic Dehn fillings $M(a,b)$,*

$$\left| \frac{\pi^2}{\Delta V(a,b)} - Q(a,b) \right| < C_2 \tag{2.4}$$

where $\Delta V(a,b) = V_0 - Vol(M(a,b))$.

Proof. (a) follows from Thurston's hyperbolic Dehn surgery theorem [**43**].

(b) The asymptotic volume formula of Neumann-Zagier [**38**] shows that the decrease in volume under Dehn filling satisfies

$$\Delta V(a,b) = \frac{\pi^2}{Q(a,b)} + O\left(\frac{1}{Q(a,b)^2}\right). \tag{2.5}$$

This implies the result of (b) when $Q(a,b)$ is sufficiently large. But there are only finitely many additional (a,b) with $M(a,b)$ hyperbolic, so the result follows. $\square$

Remark 2.3. (a) If T is a maximal embedded horospherical torus, then we can take $C_1 = 6$ by the "length 6 theorem" of Agol [**3**] and Lackenby [**31**] and the Geometrization Theorem of Perelman.

(b) The work of Hodgson-Kerckhoff [**28**, Theorem 5.12 and Figure 2], and some numerical calculations imply that

$$-7.05 \le \frac{\pi^2}{\Delta V(a,b)} - Q(a,b) \le 5.82 \tag{2.6}$$

whenever $Q(a,b) \ge 57.5041$ and $\Delta V \le 0.155$. Then by volume computations using SnapPy or Snap for the finite number of hyperbolic Dehn fillings with $\Delta V > 0.155$, an explicit value of C_2 can be determined for a given manifold M.

Numerical calculations using SnapPy suggest that $\left|\frac{\pi^2}{\Delta V(a,b)} - Q(a,b)\right|$ for hyperbolic Dehn fillings $M(a,b)$ is often considerably lower than these general bounds.

We now study the quadratic form $Q(a,b)$ for a cusped hyperbolic 3-manifold M. Let

$$S_Q = \{Q(a,b) : a, b \text{ are relatively prime integers}\}. \tag{2.7}$$

Key Idea. Let (a_0, b_0) be a pair of relatively prime integers such that $M(a_0, b_0)$ is hyperbolic and let $q_0 = Q(a_0, b_0)$. Assume that

(i) there is large enough 2-sided gap around q_0 in S_Q, and
(ii) $Q(a,b) = q_0$ has few solutions with (a,b) relatively prime integers.

Then there are few Dehn fillings $M(a,b)$ with the same volume as $M(a_0, b_0)$.

[1] Caution: this convention agrees with that used in Snap. However, in SnapPy the 'shape' reported by the function `cusp_info()` is the *complex conjugate* $\bar{\tau}$.

More precisely,

Proposition 2.4. *Let (a_0, b_0) be a pair of relatively prime integers such that $M(a_0, b_0)$ is hyperbolic and let $q_0 = Q(a_0, b_0)$. Let $C_2 = C_2(M) > 0$ be as in Proposition 2.2. Assume that*

$$\text{there exists } g > 2C_2 \text{ such that } s \in S_Q \text{ and } |s - q_0| < g \text{ implies } s = q_0.$$

Then Vol$(M(a, b))$ = Vol$(M(a_0, b_0)$ implies $Q(a, b) = Q(a_0, b_0)$. Hence the number of such manifolds $M(a, b)$ is at most $n(q_0)/2$, where $n(q_0)$ is the number of solutions to $Q(a, b) = q_0$ with a, b relatively prime integers.

Proof. Assume that $M(a, b)$ is hyperbolic with $\mathrm{Vol}(M(a, b)) = \mathrm{Vol}(M(a_0, b_0))$ and let $\Delta V = V_0 - \mathrm{Vol}(M(a_0, b_0))$. Then Proposition 2.2 implies

$$\left|\frac{\pi^2}{\Delta V} - Q(a, b)\right| < C_2 \text{ and } \left|\frac{\pi^2}{\Delta V} - Q(a_0, b_0)\right| < C_2,$$

so

$$|Q(a, b) - Q(a_0, b_0)| < 2C_2 < g.$$

Hence $Q(a_0, b_0) = Q(a, b)$ by our assumption. But (a, b) and $(-a, -b)$ Dehn fillings on M give the same manifold, so the result follows. □

Remark 2.5. More generally, there is a version of the previous Proposition which applies when the horospherical torus $T = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ has symmetries other than those induced by the maps $(a, b) \mapsto \pm(a, b)$ that extend to symmetries of the cusped manifold M.

Let G_Q be the group of integral automorphisms of the quadratic form Q,

$$G_Q = \{A \in GL(2, \mathbb{Z}) : Q(AX) = Q(X) \text{ for all } X = (x, y) \in \mathbb{Z}^2\}. \tag{2.8}$$

For each isometry g of M preserving the cusp corresponding to T, the induced action on homology gives an automorphism g_* of $H_1(T) \cong \mathbb{Z}^2$. Let $G_M \subset G_Q$ be the subgroup generated by these automorphisms g_* together the automorphism $(x, y) \mapsto (-x, -y)$. Then for each $\sigma \in G_M$, the Dehn fillings of M along (a, b) and $(a', b') = \sigma(a, b)$ give homeomorphic manifolds.

Then the previous argument shows that, with the hypotheses and notation of Proposition 2.4, the number of hyperbolic manifolds $M(a, b)$ with $\mathrm{Vol}(M(a, b)) = \mathrm{Vol}(M(a_0, b_0)$ is at most $n(q_0)/|G_M|$, where $n(q_0)$ is the number of solutions to $Q(a, b) = q_0$ with a, b relatively prime integers.

If the cusp shape parameter τ belongs to an imaginary quadratic field, then the quadratic form

$$Q(x, y) = |x + \tau y|^2 = (x + \tau y)(x + \bar{\tau} y) = x^2 + (\tau + \bar{\tau})xy + |\tau|^2 y^2$$

has rational coefficients. Hence, after rescaling, we can assume that the coefficients are relatively prime integers.

If Q is an *integral quadratic form*

$$Q(x, y) = ax^2 + bxy + cy^2. \tag{2.9}$$

where $a, b, c \in \mathbb{Z}$, then $D = b^2 - 4ac$ is called the *discriminant* of Q. The following elementary result can be found, for example, in [**33**, Theorem 201, p180], [**13**, pp25–26] or [**7**, pp49–50]. Here, and henceforth, we say that a pair of integers (x, y) is *primitive* if $\gcd(x, y) = 1$.

Lemma 2.6. *Assume that m has a primitive integer representation by an integral quadratic form Q, i.e. $Q(x_0, y_0) = m$ where $(x_0, y_0) \in \mathbb{Z}^2$ with $\gcd(x_0, y_0) = 1$. Then the discriminant D of Q is a quadratic residue modulo $4m$. In particular, every prime divisor p of m has Kronecker symbol $(D/p) = +1$.*

3. Dehn filling on the figure eight knot complement and its sister

In this section we study the closed hyperbolic manifolds obtained by Dehn filling on the figure eight knot complement and its sister, denoted $m004$ and $m003$ respectively in SnapPea notation. We will use the ideas described in the previous section to prove the following.

Theorem 3.1. *There is an infinite sequence of hyperbolic Dehn fillings $M(a_i, b_i)$ on the figure eight knot complement $M = m004$ with $a_i^2 + 12b_i^2 \to \infty$ such that if N is any orientable hyperbolic 3-manifold with $Vol(N) = Vol(M(a_i, b_i))$ then N is homeomorphic to $M(a_i, b_i)$. So the manifolds $M(a_i, b_i)$ are determined by their volumes, amongst all finite volume orientable hyperbolic 3-manifolds.*

For $m003$ and $m004$, the quadratic forms defined in equation (2.1) are closely related to the quadratic form

$$Q_0(a, b) = a^2 + ab + b^2 = |a + b\omega|^2 \tag{3.1}$$

where $\omega = \frac{1}{2}(1 - \sqrt{3}i)$ is a cube root of -1 satisfying $\omega^2 - \omega + 1 = 0$ and $\bar{\omega} = 1 - \omega$. From the symmetries of the lattice $\mathbb{Z} + \mathbb{Z}\omega$, we have

$$|a + b\omega| = |\omega^k(a + b\omega)| = |\omega^k(a + b\bar{\omega})| = |\omega^k(a + b - b\omega)|$$

for $k = 0, 1, \ldots, 5$. Hence Q_0 has symmetry group $G_{Q_0} \cong D_6$ of order 12, and the orbit of (a, b) under G_{Q_0} is

$$\{\pm(a, b), \pm(-b, a + b), \pm(-a - b, a), \pm(a + b, -b), \pm(b, a), \pm(-a, a + b)\}. \tag{3.2}$$

Now consider a maximal horospherical cusp torus for $m004$ and $m003$. In each case we choose the geometric basis for the peripheral homology used by SnapPea, consisting of two shortest simple closed geodesics. From Snap or SnapPy (or [**44**]) we have the following:

For $m004$, this torus has shortest closed geodesic of length 1, cusp shape $\tau = -2\sqrt{3}i = 4\omega - 2$ and area $2\sqrt{3}$. Hence, the (a, b) geodesic has length squared

$$Q_1(a, b) = |a + b\tau|^2 = a^2 + 12b^2 = Q_0(a - 2b, 4b). \tag{3.3}$$

This quadratic form has symmetry group $G_{Q_1} = D_2$, given by $(a, b) \mapsto (\pm a, \pm b)$, and these all extend to symmetries of $m004$, so $G_{m004} = G_{Q_1}$.

For $m003$, the torus has shortest closed geodesic of length 2, cusp shape $\tau = \omega$ and area $2\sqrt{3}$. Hence, the (a, b) geodesic has length squared

$$Q_2(a, b) = |2a + 2b\omega|^2 = 4(a^2 + ab + b^2) = 4Q_0(a, b). \tag{3.4}$$

This quadratic form has symmetry group $G_{Q_2} = G_{Q_0} = D_6$ as described above. However, $G_{m003} = D_2$ is the subgroup given by $(a, b) \mapsto \pm(a, b), \pm(a + b, -b)$. (Note that any symmetry of $m003$ must preserve the homologically trivial longitude $(-1, 2)$, hence must also preserve the orthogonal $(1, 0)$ curve.)

Proposition 3.2. *Given any integer $g > 0$ there exist infinitely many primes p such that*

(i) *p can be written as $Q_1(a,b)$ where a,b are relatively prime integers, and if $p = Q_1(a,b) = Q_1(a',b')$ then $(a',b') = (\pm a, \pm b)$ (with 4 possible choices of signs),*
(ii) *$p+k$ has no primitive integer representation as $Q_0(a,b)$ for $0 < |k| \leq g$,*
(iii) *p has no primitive integer representation as $Q_2(a,b)$.*

This follows from some elementary number theory.

Lemma 3.3. *If p is a prime congruent to 1 modulo 12, then p can be written uniquely in the form $p = Q_1(a,b) = a^2 + 12b^2$ where a,b are relatively prime positive integers.*

Proof. Euler showed that each prime $p \equiv 1 \bmod 3$ can be written uniquely in the form $p = x^2 + 3y^2$ with x,y positive integers (see for example [**13**, Chapter 1]). If $p \equiv 1 \bmod 12$, then taking congruences $\bmod 4$ shows that y is even. □

Lemma 3.4. *If p is a prime congruent to 5 modulo 6 and p divides n, then n has no representation $n = Q_0(a,b) = a^2 + ab + b^2$ with a,b relatively prime integers.*

Proof. Let p be an odd prime dividing $n = a^2 + ab + b^2$ with a,b relatively prime integers. Then by Lemma 2.6, -3 is a quadratic residue mod p. If $p \neq 3$, it follows that $p \equiv 1 \bmod 6$ by Quadratic Reciprocity (see e.g. [**25**, Thm 96]). □

Proof of Proposition 3.2. We use the previous lemmas together with the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic progressions.

For $i = 0,1,2$ let

$$S_i = \{Q_i(a,b) : a,b \text{ are relatively prime integers}\}.$$

Given any integer $g > 0$, let $p_1, p_2, \ldots, p_{2g}$ be distinct primes congruent to 5 modulo 6. Consider the integers n satisfying the congruences

$$\begin{aligned} n - i &\equiv 0 \bmod p_i && \text{for } 1 \leq i \leq g, \\ n + i &\equiv 0 \bmod p_{g+i} && \text{for } 1 \leq i \leq g, \text{ and} \\ n &\equiv 1 \bmod 12. \end{aligned}$$

By the Chinese Remainder Theorem, these have a general solution of the form

$$n \equiv n_0 \bmod 12 p_1 \ldots p_{2g}. \tag{3.5}$$

By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many solutions to (3.5) such that n is a prime, say p. For such p we have $p \equiv 1 \bmod 12$ so $p \in S_1$ by Lemma 3.3. Further $p+i$ is divisible by p_i and $p-i$ is divisible by p_{g+i} for $1 \leq i \leq g$. Hence $p+i \notin S_0, p-i \notin S_0$ for $1 \leq i \leq g$ by Lemma 3.4. Finally, $p \notin Q_2$ since p is odd. Hence p satisfies all the conditions (i)–(iii) of the proposition. □

We now combine Proposition 2.5 and Proposition 3.2 to prove the main theorem of this section.

Proof of Theorem 3.1. The quadratic forms Q_1 and Q_2 for $m004$ and $m003$ considered above give the squares of geodesic lengths on a cusp torus with area $2\sqrt{3}$. Hence the corresponding normalised quadratic forms, used in Proposition 2.2, are

$$\hat{Q}_1(a,b) = \frac{1}{2\sqrt{3}} Q_1(a,b) \text{ and } \hat{Q}_2(a,b) = \frac{1}{2\sqrt{3}} Q_2(a,b).$$

We now apply Proposition 3.2 with gap size $g = 4\sqrt{3}C$, where C is the maximum of the constants C_2 for $m004$ and $m003$ given by Proposition 2.2. This gives a sequence of primes $p_i \equiv 1 \bmod 12$ with $p_i \to \infty$ and pairs (a_i, b_i) of relatively prime integers with $p_i = a_i^2 + 12b_i^2$ satisfying the conditions of Proposition 3.2.

Then $\mathrm{Vol}(m004(a,b)) = \mathrm{Vol}(m004(a_i,b_i))$ implies that $Q_1(a,b) = Q_1(a_i,b_i)$ by Proposition 2.5, hence $(a,b) = (\pm a_i, \pm b_i)$ by condition (i) of Proposition 3.2. But these solutions differ by symmetries of Q_1 that extend to symmetries of $m004$, so $m004(a,b)$ is homeomorphic to $m004(a_i,b_i)$.

Further, $\mathrm{Vol}(m004(a_i,b_i)) = \mathrm{Vol}(m003(a,b))$ implies that $Q_1(a_i,b_i) = Q_2(a,b)$. But this has no solutions by condition (iii) of Proposition 3.2. Hence $m004(a_i,b_i)$ is uniquely determined by its volume amongst Dehn fillings on $m004$ and $m003$.

Cao and Meyerhoff [**9**] showed that the figure eight knot complement $m004$ and its sister $m003$ are the unique orientable cusped hyperbolic 3-manifolds of smallest volume v_ω. The work of Jørgensen and Thurston (see [**43**], [**24**]) shows that any sequence of closed hyperbolic 3-manifolds with volume approaching v_ω from below has a subsequence whose geometric limit is a cusped hyperbolic 3-manifold M_∞ with volume v_ω, and that the manifolds in the subsequence are obtained by Dehn filling on M_∞. It follows that there exists $\epsilon > 0$ such that if N is any closed orientable hyperbolic 3-manifold with $v_\omega - \epsilon < \mathrm{Vol}(N) < v_\omega$ then N can be obtained by Dehn filling on $m003$ or $m004$. Now the result follows from the previous observations. □

4. Dehn filling on the Whitehead link complement and its sister

We now study the 1-cusped hyperbolic 3-manifolds obtained by Dehn filling on the Whitehead link complement and its sister, the complement of the $(-2,3,8)$-pretzel link, denoted $m129$ and $m125$ respectively in SnapPea notation. Note that each of these manifolds has a symmetry interchanging the cusps, so it suffices to consider Dehn fillings on cusp 0. See also [**27, 37**] and [**1**] for detailed discussions of the geometry and topology of these manifolds and their Dehn fillings.

The main result of this section is the following:

Theorem 4.1. *There is an infinite sequence of hyperbolic Dehn fillings $M(a_i,b_i)$ on one cusp of the $(-2,3,8)$-pretzel link complement $M = m125$ with $a_i^2 + b_i^2 \to \infty$ such that if N is any orientable cusped hyperbolic 3-manifold with Vol(N) = Vol($M(a_i,b_i)$) then N is homeomorphic to $M(a_i,b_i)$. So the manifolds $M(a_i,b_i)$ are determined by their volumes, amongst all orientable cusped hyperbolic 3-manifolds.*

The proof is similar to the proof of Theorem 3.1. However there is an extra difficulty because not every symmetry of the cusp torus extends to a symmetry of the manifold $m125$. To deal with this, we look at the next terms in the Neumann-Zagier asymptotic expansion for volume change during Dehn filling.

The quadratic forms for $m125$ and $m129$ given by (2.1) are closely related to the quadratic form

$$Q_0(a,b) = a^2 + b^2 = |a + bi|^2. \tag{4.1}$$

From the symmetries of the lattice $\mathbb{Z} + \mathbb{Z}i$, we have

$$|a+bi| = |i^k(a+bi)| = |i^k(a-bi)|$$

for $k = 0, 1, \ldots, 3$, and Q_0 has symmetry group $G_{Q_0} \cong D_4$ of order 8, with the orbit of (a,b) under G_{Q_0} given by

$$\{\pm(a,b), \pm(-b,a), \pm(a,-b), \pm(b,a)\}. \tag{4.2}$$

Consider a maximal horospherical cusp torus for $m129$ and $m125$. In each case we choose the geometric basis for peripheral homology used by SnapPea. From SnapPea (or [**27, 37, 1**]) we have the following:

For $m125$, the torus has cusp shape $\tau = i$. If we normalise so that the torus has area 2 then the (a,b) geodesic has length squared

$$Q_1(a,b) = 2|a+bi|^2 = 2(a^2+b^2) = 2Q_0(a,b). \tag{4.3}$$

This quadratic form has symmetry group $G_{Q_1} = G_{Q_0} = D_4$ as described above. The subgroup of symmetries of Q_1 extending to symmetries of $m125$ is the subgroup of orientation preserving symmetries $G_{m125} = C_4$, given by $(a,b) \mapsto \pm(a,b), \pm(-b,a)$.

For $m129$, the torus has cusp shape $\tau = 2i$. If we normalise so that the torus has area 2 then the (a,b) geodesic has length squared

$$Q_2(a,b) = |a+2bi|^2 = (a^2+4b^2) = Q_0(a,2b). \tag{4.4}$$

This quadratic form has symmetry group $G_{Q_2} = D_2$ consisting of $(a,b) \mapsto (\pm a, \pm b)$, but $G_{m129} = C_2$ is the subgroup $(a,b) \mapsto \pm(a,b)$.

Proposition 4.2. *For each $g > 0$ there exist infinitely many integers $m = 2p$, with p prime, such that*

(i) *m can be written as $Q_1(a,b)$ where a, b are relatively prime integers, and if $Q_1(a,b) = Q_1(a',b')$ then $(a',b') = (\pm a, \pm b)$ or $(a',b') = (\pm b, \pm a)$.*
(ii) *$m + k$ has no primitive integer representation as $Q_0(a,b)$ for $0 < |k| \leq g$,*
(iii) *m has no primitive integer representation as $Q_2(a,b)$.*

This again follows from some elementary number theory.

Lemma 4.3. *If p is a prime congruent to 1 modulo 4, then p can be written in the form $p = a^2 + b^2$ where a, b are relatively prime positive integers. Further, if $p = (a')^2 + (b')^2$ where a', b' are positive integers then $(a',b') = (a,b)$ or (b,a).*

Proof. This is a result of Euler (see, for example, [**25**, Chapter 15]). □

Lemma 4.4. *If p is a prime congruent to 3 modulo 4 and p divides n, then n has no representation $n = Q_0(a,b) = a^2 + b^2$ with a, b relatively prime integers.*

Proof. Let p be an odd prime dividing $n = a^2 + b^2$ with a, b relatively prime integers. Then by Lemma 2.6, -1 is a quadratic residue mod p. It follows that $p \equiv 1 \bmod 4$ (see e.g. [**25**, Thm 82]). □

Lemma 4.5. *If $n \equiv 2 \mod 4$ then $n = Q_2(a,b) = a^2 + 4b^2$ has no solution with a, b integers.*

Proof. Reduce modulo 4. □

Proof of Proposition 4.2. This follows from the previous lemmas together with the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic progressions, as in the proof of Proposition 3.2. □

Proof of Theorem 4.1. The quadratic forms Q_1 and Q_2 for $m125$ and $m129$ considered above give the squares of geodesic lengths on a cusp torus with area 2. Hence the corresponding normalised quadratic forms are given by

$$\hat{Q}_1(a,b) = \frac{1}{2}Q_1(a,b) \text{ and } \hat{Q}_2(a,b) = \frac{1}{2}Q_2(a,b).$$

We now apply Proposition 3.2 with gap size $g = 4C$, where C is the maximum of the constants C_2 for $m125$ and $m129$ given by Proposition 2.2.

This gives a sequence of primes $p_i \equiv 1 \bmod 4$ with $p_i \to \infty$ and pairs (a_i, b_i) of relatively prime integers with $p_i = a_i^2 + b_i^2$ satisfying the conditions of Proposition 4.2. Then $\mathrm{Vol}(m125(a,b)) = \mathrm{Vol}(m125(a_i,b_i))$ implies that $Q_1(a,b) = Q_1(a_i,b_i)$, hence $(a,b) = \pm(a_i,b_i), \pm(-b_i,a_i), \pm(a_i,-b_i)$ or $\pm\,(b_i,a_i)$.

Now the surgeries $\pm(a,b), \pm(-b,a)$ give manifolds which are homeomorphic, hence have the same volume. Similarly, the surgeries $\pm(a,-b), \pm(b,a)$ give homeomorphic manifolds of the same volume. Next, we use the degree 4 terms in the Neumann-Zagier asymptotic formula to show that $\mathrm{Vol}(m125(a,b)) \neq \mathrm{Vol}(m125(b,a))$ for all sufficiently large (a,b).

In [**1**], Aaber and Dunfield study the Neumann-Zagier asymptotic expansion of volume for Dehn fillings on the Whitehead link sister $m125$. Using their calculations we obtain the following asymptotic formula for decrease in volume for surgeries on cusp 0 of $m125$ (see Example 5.3 and equation (5.11) below):

$$\Delta V_{m125}(a,b) = \frac{\pi^2}{a^2+b^2} - \frac{\pi^4(a^4 - 12a^3b - 6a^2b^2 + 12ab^3 + b^4)}{24\,(a^2+b^2)^4} + O\left(\frac{1}{(a^2+b^2)^3}\right).$$

Hence, we have

$$\Delta V_{m125}(a,b) - \Delta V_{m125}(b,a) = \frac{\pi^4 ab\left(a^2-b^2\right)}{(a^2+b^2)^4} + O\left(\frac{1}{(a^2+b^2)^3}\right). \tag{4.5}$$

Lemma 4.6. *If a and b are integers with $a > b > 0$, then $ab(a^2-b^2) \geq r^3/4$, where $r^2 = a^2+b^2$.*

Proof. Consider the function

$$f(a,b) = ab(a^2-b^2) = ab(a+b)(a-b).$$

Now we have

(i) If $b \leq a/2$ then $b(a-b) \geq b(a-a/2) = ba/2 \geq a/2$, since $b \geq 1$.
(ii) If $b \geq a/2$ then $b(a-b) \geq a/2 \cdot (a-b) \geq a/2$, since $a-b \geq 1$.

Further, $a^2+b^2 = r^2$ and $a > b > 0$ imply that $a^2 \geq r^2/2$ and $a+b \geq r$. Hence

$$f(a,b) = a(a+b)b(a-b) \geq a(a+b)\cdot a/2 = a^2(a+b)/2 \geq r^3/4,$$

as desired. □

Equation (4.5) shows that, whenever a^2+b^2 is sufficiently large,

$$\Delta V_{m125}(a,b) - \Delta V_{m125}(b,a) \geq \frac{\pi^4 ab\left(a^2-b^2\right) - C(a^2+b^2)}{(a^2+b^2)^4},$$

where C is a positive constant. Hence, by Lemma 4.6, there exists a constant $R > 0$ such that if (a,b) are relatively prime integers with $a > b > 0$ and $a^2+b^2 \geq R^2$, then $\Delta V_{m125}(a,b) > \Delta V_{m125}(b,a)$.

This completes the proof that for all sufficiently large (a_i, b_i), the Dehn fillings $m125(a_i, b_i)$ are determined by their volumes amongst Dehn fillings on $m125$.

Further, $\mathrm{Vol}(m125(a_i, b_i)) = \mathrm{Vol}(m129(a, b))$ implies that $Q_1(a_i, b_i) = Q_2(a, b)$, which has no solutions by condition (iii) of Proposition 4.2. Hence, for all sufficiently large (a_i, b_i), the manifolds $m125(a_i, b_i)$ are determined by their volumes amongst Dehn fillings on $m129$ and $m125$.

The result now follows as in the proof of Theorem 3.1 since $m129$ and $m125$ are the unique 2-cusped manifolds of volume v_{ω^2} by the work of Agol [**4**]. So any 1-cusped manifold with volume less than v_{ω^2} but sufficiently close to v_{ω^2} comes from Dehn filling on $m129$ or $m125$. $\square$

5. Higher order terms in the Neumann-Zagier asymptotic formula

Let M be a cusped hyperbolic 3-manifold with horospherical torus cross section T, and chose meridian and longitude generators for $H_1(T) = \mathbb{Z}^2$ as in section 2. Let u, v denote the logarithms of the holonomies of the meridian and longitude respectively, chosen so that $u = v = 0$ at the complete hyperbolic structure on M. Then $-v$ has a power series expansion

$$-v = \sum_{n=1}^{\infty} c_n u^n, \tag{5.1}$$

where $c_n = 0$ for all even n and $c_1 = -\tau$ where τ is our cusp shape parameter.[2]

Then Neumann and Zagier [**38**, equation (62), p328]) derive the following asymptotic formula for the decrease in volume ΔV during Dehn filling for (p, q) Dehn filling on one cusp T of the cusped hyperbolic 3-manifold M:

$$\Delta V = \frac{|\mathrm{Im}\,(c_1)|\pi^2}{|z|^2} - 2\pi^4\,\mathrm{Im}\,\left[\frac{c_3}{z^4}\right] + O\left(\frac{1}{|z|^6}\right), \tag{5.2}$$

where $z = p + q\tau$.

If we write $z = re^{i\theta}$, then this becomes

$$\Delta V = \frac{|\mathrm{Im}\,(\tau)|\pi^2}{r^2} - \frac{2\pi^4}{r^4}\mathrm{Im}\,\left[c_3 e^{-4i\theta}\right] + O\left(\frac{1}{r^6}\right). \tag{5.3}$$

We now give some calculations for the manifolds studied in the previous sections.

Example 5.1. For $m004$, Neumann and Zagier ([**38**, p331]) show that $c_1 = 2\sqrt{-3} = -\tau$ and $c_3 = \frac{2\sqrt{-3}}{3}$, and obtain the following asymptotic formula:

$$\Delta V_{m004}(p, q) = \frac{2\sqrt{3}\pi^2}{p^2 + 12q^2} - \frac{4\sqrt{3}(p^4 - 72p^2q^2 + 144q^4)\pi^4}{3(p^2 + 12q^2)^4} + O\left(\frac{1}{(p^2 + 12q^2)^3}\right). \tag{5.4}$$

Writing $z = p + \tau q = re^{i\theta}$ as above, this simplifies to

$$\Delta V_{m004} = \frac{2\sqrt{3}\pi^2}{r^2} - \frac{4\pi^4\cos(4\theta)}{\sqrt{3}r^4} + O\left(\frac{1}{r^6}\right). \tag{5.5}$$

[2] In [**38**], a non-standard orientation convention for meridian and longitude was used; this was pointed out in [**37**]. Replacing the basis in [**38**] by *meridian*, $-$(*longitude*) corresponding to $u, -v$ gives our version of the Neumann-Zagier formula.

Example 5.2. Let WL denote the Whitehead link complement as drawn in Figure 2 (and in [**27**], [**37**]), with the standard topological choice of meridians and longitudes. Then Hodgson-Meyerhoff-Weeks [**27**, Figure 8 with $p = q = 1$] show that the hyperbolic Dehn fillings $WL(1,1)(m,l)$ and $WL(-5,1)(m,-l-m/2)$ have equal volumes since they have homeomorphic 2-fold covers, whenever m,l are relatively prime integers with m a multiple of 4.

Now $WL(1,1)$ and $WL(-5,1)$ are homeomorphic to $m004$ and $m003$ respectively. In fact, using SnapPy or Snap, there are isometries taking $WL(1,1)(m,l)$ to $m004(m,l)$ and $WL(-5,1)(m,-l-m/2)$ to $m003(m/2-l,2l)$. Hence we have

$$\mathrm{Vol}(m004(m,l)) = \mathrm{Vol}(m003(a,b)) \tag{5.6}$$

for hyperbolic Dehn fillings such that $(a,b) = (m/2-l,2l)$ and $(m,l) = (2a+b,b/2)$ for pairs of relatively prime integers with m a multiple of 4.

Since the volume is a real analytic function on hyperbolic Dehn surgery space (using u as coordinate, as in [**38**]), it follows that (5.6) holds for all large real values of (p,q). Hence we can deduce the asymptotic formula for decrease in volume for $m003$:

$$\begin{aligned}\Delta V_{m003}(a,b) &= \Delta V_{m004}(2a+b,b/2)\\ &= \frac{\sqrt{3}\pi^2}{2\left(a^2+ab+b^2\right)} - \frac{\pi^4\left(-18b^2(2a+b)^2+(2a+b)^4+9b^4\right)}{64\sqrt{3}\left(a^2+ab+b^2\right)^4} + \dots\end{aligned} \tag{5.7}$$

Introducing polar coordinates $re^{i\theta} = a + b\omega$, this simplifies to

$$\Delta V_{m003}(a,b) = \frac{\sqrt{3}\pi^2}{2r^2} - \frac{\pi^4}{4\sqrt{3}}\frac{\cos(4\theta)}{r^4} + O\left(\frac{1}{r^6}\right). \tag{5.8}$$

Example 5.3. For the Whitehead link sister $m125$, Aaber and Dunfield [**1**] study the Neumann-Zagier asymptotic expansion of volume for Dehn fillings. If u,v denote the logarithms of holonomies of the meridian and longitude for the geometric basis for cusp 0 of $m125$, then they show [**1**, p1019] that

$$-v = iu + \frac{-3+i}{48}u^3 + \dots \tag{5.9}$$

so $c_1 = -\tau = i$ and $c_3 = \frac{-3+i}{48}$ in our notation. This gives the following asymptotic formula for decrease in volume for surgeries on cusp 0:

$$\Delta V_{m125}(p,q) = \frac{\pi^2}{|z|^2} - 2\pi^4\,\mathrm{Im}\left[\frac{-3+i}{48z^4}\right] + O\left(\frac{1}{|z|^6}\right), \tag{5.10}$$

where $z = p + q\tau = p - qi$.

More explicitly:

$$\Delta V_{m125}(p,q) = \frac{\pi^2}{p^2+q^2} - \frac{\pi^4(p^4-12p^3q-6p^2q^2+12pq^3+q^4)}{24\left(p^2+q^2\right)^4} + O\left(\frac{1}{(p^2+q^2)^3}\right). \tag{5.11}$$

Example 5.4. For the Whitehead link complement WL (as drawn in Figure 2) we can compute the asymptotic expansion for ΔV using the work of Neumann-Reid [**37**]. Let u,v denote the logarithms of holonomies of the standard meridian and longitude for the link complement. Then [**37**] shows that

$$u = \log x + \log(x+1) - \log(x-1) \text{ and } v = 4\log x - 2\pi i,$$

where $x \in \mathbb{C}$ is a suitable simplex parameter. From these equations we obtain

$$e^u = \frac{x(x+1)}{(x-1)} \text{ where } x = ie^{-v/4}.$$

This gives a quadratic equation for x with the relevant solution given by

$$x = \frac{1}{2}\left(\sqrt{-6e^u + e^{2u} + 1} + e^u - 1\right) = ie^{-v/4},$$

and so

$$\begin{aligned} -v &= -4\log\left(\frac{-i}{2}\left(\sqrt{-6e^u + e^{2u} + 1} + e^u - 1\right)\right) \\ &= (-2+2i)u + \frac{iu^3}{6} + O(|u|^5). \end{aligned} \tag{5.12}$$

Hence $c_1 = -\tau = (-2+2i)$ and $c_3 = \frac{i}{6}$ in our previous notation.

This gives the following asymptotic formula for decrease in volume for Dehn filling on one cusp of WL:

$$\Delta V_{WL}(p,q) = \frac{2\pi^2}{|z|^2} - 2\pi^4 \operatorname{Im}\left[\frac{i}{6z^4}\right] + O\left(\frac{1}{|z|^6}\right), \tag{5.13}$$

where $z = p + q\tau = p + q(2-2i)$. Hence

$$\Delta V_{WL}(p,q) = \frac{2\pi^2}{p^2+4pq+8q^2} - \frac{\pi^4\left(p^2-8q^2\right)\left(p^2+8pq+8q^2\right)}{3\left(p^2+4pq+8q^2\right)^4} + \dots. \tag{5.14}$$

We can convert from the standard peripheral curves for WL to the geometric peripheral curves on $m129$ using: $WL(p,q) = m129(a,b)$ where $a = p+2q, b = -q$ or $p = a+2b, q = -b$. Then the new cusp shape parameter for $m129$ is $\tau' = -2i$ and

$$\begin{aligned} \Delta V_{m129}(a,b) &= \Delta V_{WL}(a+2b,-b) \\ &= \frac{2\pi^2}{a^2+4b^2} - \frac{\pi^4\left(a^4 - 24a^2b^2 + 16b^4\right)}{3\left(a^2+4b^2\right)^4} + \dots \end{aligned} \tag{5.15}$$

6. Hyperbolic graphs with parabolic meridians

In this section, we recall some results on hyperbolic graphs in S^3 with parabolic meridians; for details see [**26**] and [**34**]. In this paper, spatial graphs may contain link components, and we regard every link as a spatial graph without vertices. We use the word *link* to emphasize that the graph does not have any vertices. Let G be a spatial trivalent graph in S^3 and let N be a manifold obtained from $S^3 \setminus G$ by removing an open regular neighbourhood of each vertex. Then N is a manifold with boundary consisting of thrice punctured spheres, one corresponding to each vertex of G. We say that G is *hyperbolic* if N admits a hyperbolic metric of finite volume with geodesic boundary; then the meridians of the graph correspond to parabolic isometries. We can construct a hyperbolic link from a hyperbolic graph by using the following.

Lemma 6.1 ([**34**])**.** *For hyperbolic graphs, the moves shown in Figure 5 are volume preserving, where regular neighbourhoods of the two trivalent vertices have been removed in middle diagram.*

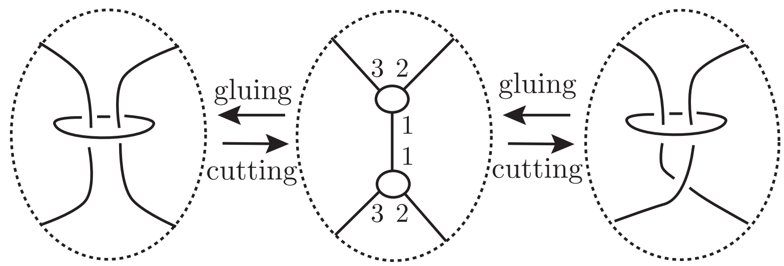

FIGURE 5. Volume preserving moves on hyperbolic graphs

Proof. Note that each vertex in a hyperbolic graph corresponds to a totally geodesic thrice punctured sphere and each edge corresponds to an annulus cusp. Moreover, the hyperbolic structure on a thrice punctured sphere is unique by [**2**]; hence any orientation preserving homeomorphism between hyperbolic thrice punctured spheres is isotopic to an isometry. Since any homeomorphism between thrice punctured spheres is uniquely determined by its action on the cusps, we may denote homeomorphisms as elements of S_3, the symmetry group of degree 3.

Fix labels for the cusps of the thrice punctured spheres as in Figure 5. If we glue the boundaries via the homeomorphism corresponding to the identity permutation, then we get the tangle on the left of Figure 5. If we glue the boundaries via the homeomorphism corresponding to the permutation interchanging 2 and 3, we get the tangle on the right of Figure 5.

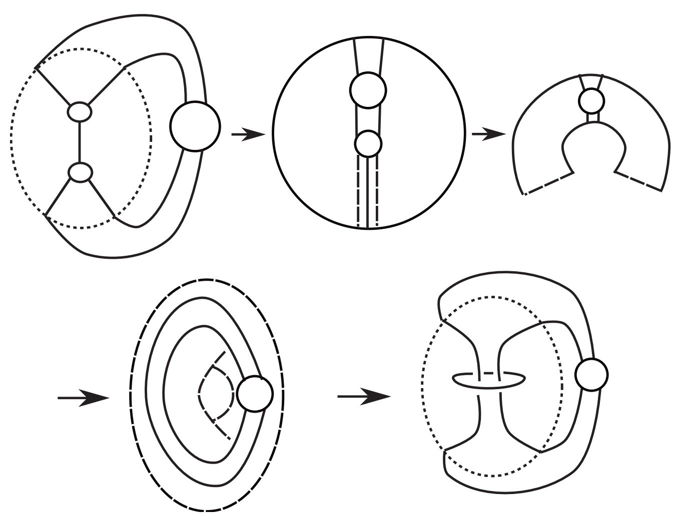

FIGURE 6. Gluing by a homeomorphism

This is illustrated in Figure 6. Here we regard S^3 as $B^3 \cup B^3$ where one ball B^3 is a regular neighbourhood of a vertex of the graph shown in the middle of Figure 5. We then reverse the inside and the outside with respect to the boundary sphere $S^2 = \partial B^3$. After gluing, we get a link or graph in $S^2 \times S^1$. The component coming from the cusp labelled by 1 is a loop which corresponds to a generator of

the fundamental group of $S^2 \times S^1$, and its complement is homeomorphic to a solid torus. Thus we get a tangle as shown on the left or right of Figure 5.

The inverse move (left or right to middle) is valid since any essential thrice punctured sphere (or, 2-punctured disk) in a hyperbolic 3-manifold is totally geodesic ([**2**]). Since we are dealing with hyperbolic graphs, the 2-punctured disk on the left or right of Figure 5 is essential (see [**39**] Lemma 2.1). □

Definition 6.2. A *fully augmented link* is a link obtained by applying the gluing moves of Lemma 6.1 to a planar diagram of a hyperbolic graph (embedded in the plane) so that the resulting graph is actually a link. A *crossing circle* is a component of a fully augmented link which appears as a circle in Figure 5. A *knot component* of a fully augmented link is a component which is not a crossing circle.

Remark 6.3. These definitions of a fully augmented link, a crossing circle, and a knot component are equivalent to the original definitions in [**20**] or [**39**].

6.1. Polyhedral decomposition for the complement of a hyperbolic graph with a planar diagram. Given a hyperbolic graph P with a planar diagram D, let V be the set of vertices of P and $N_P = S^3 \setminus P \setminus (\cup_{v \in V} \mathcal{N}(v))$ where $\mathcal{N}(v)$ is a regular neighbourhood of v. Let Π be the plane containing D. Then N_P admits hyperbolic metric of finite volume with geodesic boundary. We can decompose N_P into two isometric convex ideal hyperbolic polyhedra by the following procedure, depicted in Figure 7.

Step 1: Cut N_P along Π.
Step 2: Collapse each edge of P to a point.

Let L be a fully augmented link which is obtained from D by the gluing moves of Lemma 6.1. Then this decomposition is exactly equal to the decomposition of $S^3 \setminus L$ found by Agol-Thurston in [**32**] (see also [**20**], [**39**]).

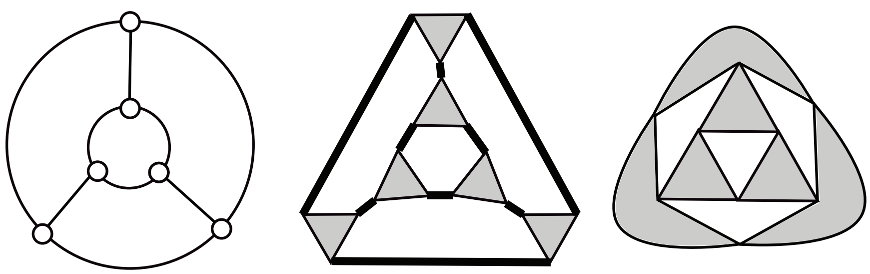

FIGURE 7. Polyhedral decomposition of a graph complement

We shade the faces coming from the thrice punctured sphere boundaries of N_P and leave the faces on Π white as in [**20**], [**39**]. Note that every shaded face is an ideal triangle, and since reflection ϕ about Π leaves each thrice punctured sphere boundary invariant, the shaded faces are all orthogonal to the adjacent white faces. Hence the dihedral angle of each edge is $\pi/2$. Since ϕ preserves white faces, each face of these polyhedra is totally geodesic. Hence each white face extends to the boundary at infinity S^2_∞ of hyperbolic space to give a Euclidean circle and such circles corresponding to adjacent faces are tangent to each other. Therefore the white faces of each polyhedron determine a circle packing of S^2_∞ whose nerve is isomorphic to the dual of the original graph diagram D.

7. Associated circle packings and cusp moduli

In this section we study the circle packing corresponding to the hyperbolic graph W_n shown in Figure 8.

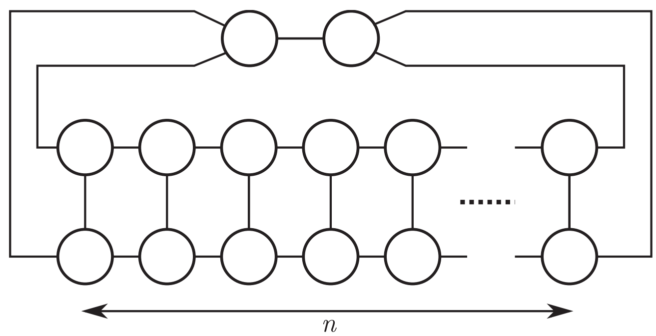

FIGURE 8. Graph W_n with $2n+2$ vertices

The two isometric polyhedra in the decomposition of the complement of W_n described in section 6 are obtained by gluing together n regular ideal octahedra (see Figure 9 (right)). Therefore the volume of its complement is $2nV_8$ where $V_8 = 3.663862\ldots$ is the volume of the regular ideal octahedron. A circle packing corresponding to the graph is depicted in Figure 9 (left). Each tangency point of the circle packing corresponds to an annulus cusp of the complement of W_n.

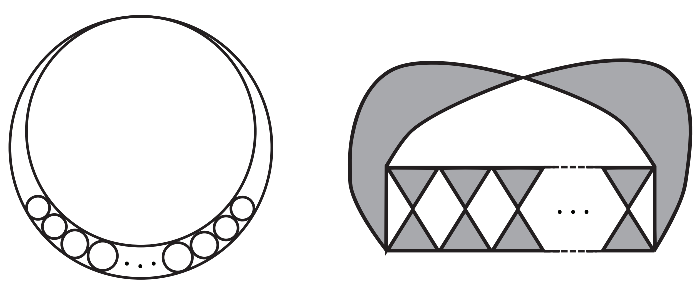

FIGURE 9. A circle packing obtained from the graph W_n (left) and a (topological) polyhedral decomposition (right)

Recall from Remark 2.1 that each torus cusp of a hyperbolic 3-manifold has a Euclidean structure $\mathbb{C}/\Gamma$ where $\Gamma \cong \mathbb{Z}^2 \subset \mathbb{C}$ is a lattice, and its shape parameter is determined by giving two independent translations corresponding to a choice of basis for Γ. Since there are infinitely many ways to choose such a basis, this shape parameter is not unique. The *modulus* of a torus cusp is defined as the complex ratio

$$z = (\text{second shortest translation})/(\text{shortest translation}). \tag{7.1}$$

It is a complex number lying in the region $|\mathrm{Re}(z)| \leq 1/2$ and $|z| \geq 1$. If z lies on the boundary of this region, we choose it so that $\mathrm{Re}(z) \geq 0$.

For the complement of W_n, there are three different types of annulus cusps, see Figures 10, 11 and 12. The circle packing diagrams giving the cusp shapes

are obtained by Möbius transformations which map one of the marked cusps to infinity. Since the graph complement is reconstructed by gluing the two isometric polyhedra along their corresponding white faces, we get the boundary of a horoball neighbourhood of the annulus cusp by gluing two copies of rectangles together along corresponding edges depicted by solid lines. Later, we glue these annuli cusps together along their boundaries (depicted as dotted lines) to get a torus cusp, and compute its modulus.

By using Lemma 6.1, we obtain hyperbolic links from W_n. The modulus of each cusp can be computed by knowing how to glue the rectangles according to the gluing pattern of thrice punctured spheres.

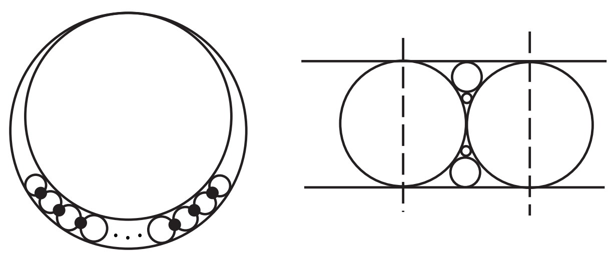

Figure 10. Cusp modulus for ideal vertices of type 1

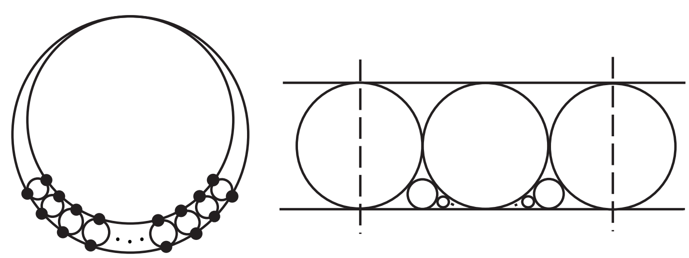

Figure 11. Cusp modulus for ideal vertices of type 2

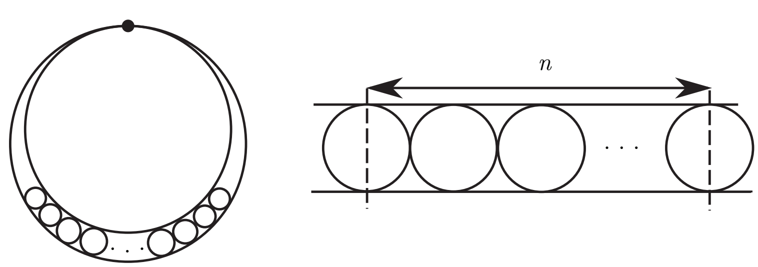

Figure 12. Cusp modulus for ideal vertices of type 3

Example 7.1. The graph $G_{n,k}$ in Figure 13 (top) is obtained from the graph in Figure 8 by $k+2$ applications of Lemma 6.1, where $k+2 \leq n$. The torus cusp corresponding to the knot component (with k crossings) comes from the $2(k+1)$ marked vertices in Figure 13 (bottom). Note that in Figure 13 (also in Figure 15), the choice of crossings below the crossing circles is not important, because changing an over crossing to an under crossing does not change the graph complement.

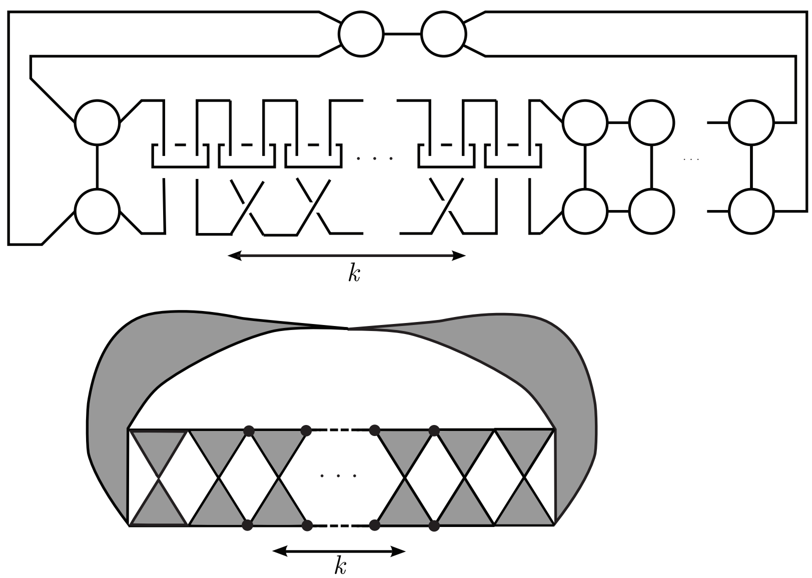

FIGURE 13. A graph $G_{n,k}$ obtained from W_n (top) and vertices corresponding to the knot component cusp (bottom)

In Figure 14, we depict the shapes of the torus cusps. The left diagram shows the shape of a crossing circle cusp without a half twist below, the middle shows the shape of a crossing circle cusp with a half twist, and the right shows the shape of the cusp which comes from the knot component.

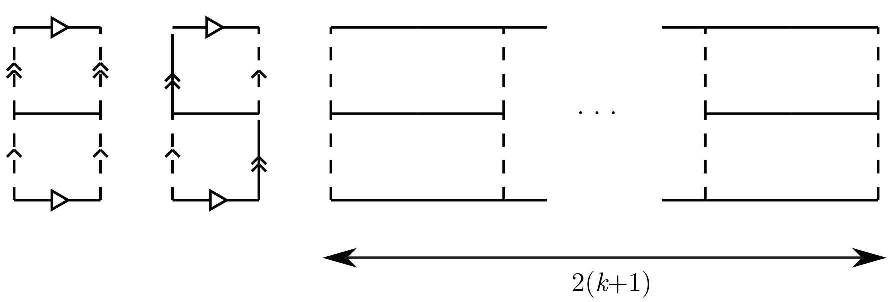

FIGURE 14. Shapes of the torus cusps for the graph $G_{n,k}$ in Figure 13

The shape of the torus cusp of the knot component in Figure 13 is obtained by arranging $2 \times 2(k+1)$ copies of the 1×2 rectangle in Figure 11 to form a $2 \times 4(k+1)$ rectangle with opposite edges identified by translations. Therefore, its

cusp modulus is $2(k+1)\sqrt{-1}$. The modulus of each crossing circle cusp depends on the existence of a half twist; if it has a half twist below, the modulus is $\sqrt{-1}$ and if not, it is $2\sqrt{-1}$. (Note that these moduli do not depend on the total number of vertices in the graph $G_{n,k}$.)

8. Construction of different manifolds sharing the same volume

We start with the graph with $2n$ vertices shown in Figure 15. This graph is obtained by applying the gluing move in Lemma 6.1 $n+1$ times to W_{2n}.

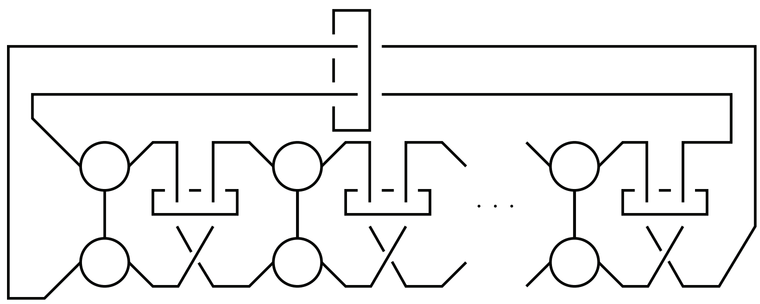

FIGURE 15. Starting point: graph with $2n$ vertices (represented by circles)

First, we label every remaining pair of thrice punctured spheres by an element of $\{0,1\}$. For each such pair we apply Lemma 6.1 as follows. We glue the pair of thrice punctured spheres labelled by 0 (resp. 1) without (resp. with) a half twist, as in Figure 5 (left) (resp.(right)). By this construction, we get a hyperbolic link complement M_b for each cyclic binary word b of length n and its volume is $4nV_8$.

Now we consider the decomposition of M_b into isometric convex hyperbolic polyhedra with shaded faces described in section 6.1. Note that every edge lies in exactly one shaded triangular face, since the faces have a checkerboard colouring.

Definition 8.1 ([**20**], [**39**])**.** Given an edge e in such a polyhedral decomposition, let t be the shaded ideal triangle containing e. Then the *midpoint* m of e is the foot of the perpendicular in t from the vertex of t opposite e to e.

Since isometries preserve angles, when we glue two thrice punctured sphere boundaries together by an isometry, it maps midpoints to midpoints. Moreover, for fully augmented links, we may expand horoball neighbourhoods of the cusps to find a horoball packing such that the horoballs bump together precisely at the midpoints of edges. More precisely,

Theorem 8.2 ([**20**], [**39**])**.** *Let L be a fully augmented link. Then there exist horoball neighbourhoods of the cusps of $S^3 \setminus L$ such that the midpoint of every edge is a point of tangency of the corresponding horospherical tori.*

Remark 8.3. For each M_b, we fix the horoball packing $\mathcal{H}_b$ given by Theorem 8.2. In fact, this horoball packing depends only on the geometry of M_b and not on any choice of polyhedral decomposition for M_b. This follows since the horospherical tori for $\mathcal{H}_b$ can be defined as follows: For each cusp with modulus $\sqrt{-1}$ or $2\sqrt{-1}$ we choose the horospherical torus of area 2, and for each cusp with modulus $m\sqrt{-1}$ with $m > 2$ we choose the horospherical torus of area $4m$. These areas are easily

calculated from the diagrams in Figures 10–12 and Figure 14, using the fact that each dotted edge in Figures 10-12 has length 1 in the horospherical tori for $\mathcal{H}_b$, (since it is a horocycle joining midpoints of edges in an ideal hyperbolic triangle).

Theorem 8.4. *Let b_i be a binary cyclic word of length $n \geq 3$ and let M_{b_i} be the hyperbolic link complement obtained from b_i by the above procedure for $i = 1, 2$. Then M_{b_1} is isometric to M_{b_2} if and only if b_1 and b_2 are related by the natural action of D_n, the dihedral group of order $2n$.*

Proof. Let b be a binary cyclic word of length n. If b contains $k \geq 1$ zeros, then we can split b into k subwords of the form $p_i = 011 \cdots 10$ with i 1's ($i \geq 0$) by using each zero in b exactly twice.

After the gluing move in Lemma 6.1, each p_i produces a torus cusp with $2i+1$ crossings, whose modulus is $m_i = 2(2i+2)\sqrt{-1} = 4(i+1)\sqrt{-1}$ (see Example 7.1). Let $\{p_{i_1}, p_{i_2}, ..., p_{i_k}\}$ be the decomposition of b, and let C_j be the cusp corresponding to the knot component produced by p_{i_j}. We get a cyclic sequence $\{m_{i_1}, m_{i_2}, ..., m_{i_k}\}$ of cusp moduli from b. Note that the total number of cusps of M_b is $2n+1+k$; the crossing circle cusp C at the top of Figure 15 has modulus $n\sqrt{-1}$ (from the cusp diagram in Figure 12), and the other $2n$ crossing circle cusps have modulus $\sqrt{-1}$ or $2\sqrt{-1}$ by Example 7.1.

We now only consider cusps whose modulus has norm greater than 2. For $n \geq 3$, these are the cusps $C_1, \ldots C_k$ and C. From M_b, we will construct a labelled *cusp graph* $\mathcal{C}_b$ with vertices $\{v_1, v_2, ..., v_{k+1}\}$ as follows. We label v_j by m_{i_j} for $1 \leq j \leq k$ and v_{k+1} by $n\sqrt{-1}$, the modulus of C. Then join v_i and $v_{i'}$ by an edge if in $\mathcal{H}_b$, horoball neighbourhoods of the cusps corresponding to v_i and $v_{i'}$ are tangent to each other. In $\mathcal{H}_b$, the horoball neighbourhood of each cusp C_j bumps only into C_{j-1}, C_{j+1} (with index modulo k), and C. Therefore, the graph $\mathcal{C}_b$ consists of a cycle of length k whose vertices are labelled cyclically by the m_{i_j}'s and one vertex labelled by n joined to every vertex of the cycle.

The special case when $b = 111 \cdots 1$ and $k = 0$ is slightly different. Then M_b has one cusp C of modulus $n\sqrt{-1}$ as above, two knot components giving cusps C_1, C_2 of modulus $2n\sqrt{-1}$, and $2n$ cusps of modulus $\sqrt{-1}$. In this case, the cusp graph $\mathcal{C}_b$ is a 3-cycle.

Conversely given such a labelled cusp graph, we can reconstruct a binary cyclic word. Therefore given two binary cyclic words b_1 and b_2 of length n, the graph $\mathcal{C}_{b_1}$ is isomorphic to $\mathcal{C}_{b_2}$ as a graph with labelled vertices if and only if b_1 is related to b_2 by the natural action of D_n. Moreover, by Remark 8.3, any isometry between manifolds M_{b_1} and M_{b_2} takes the horoball packing $\mathcal{H}_{b_1}$ to $\mathcal{H}_{b_2}$. Hence, the manifold M_{b_1} is isometric to M_{b_2} if and only if the graph $\mathcal{C}_{b_1}$ is isomorphic to $\mathcal{C}_{b_2}$ as a graph with labelled vertices. This completes the proof. □

Theorem 8.5. *For each $n \geq 3$, there are at least $2^n/(2n)$ different hyperbolic link complements of volume $4nV_8$.*

Proof. By theorem 8.4, there are at least as many hyperbolic link complements of volume $4nV_8$ as cyclic binary words of length n, up to the action of D_n. But there are 2^n based cyclic binary words of length n, and each orbit under the D_n action has at most $2n$ elements. So the result follows. □

Remark 8.6. For the sequence $v_n = 4nV_8$, this gives a logarithmic growth rate

$$\limsup_{n\to\infty} \frac{\log N_L(v_n)}{v_n} \geq \frac{\log 2}{4V_8} \approx 0.0472962.$$

Chesebro-BeBlois [**11**] construct examples of $2^n/2$ hyperbolic link complements of volume $w_n \approx 24.092184n + 2V_8$, giving a logarithmic growth rate

$$\limsup_{n\to\infty} \frac{\log N_L(w_n)}{w_n} \geq 0.0287706.$$

9. Some Open Questions

(1) Find additional exact values of $N(v)$ or upper bounds on $N(v)$.

(2) (Gromov [**24**], 1979) Is $N(v)$ locally bounded?

This reduces to the question of whether the number of manifolds with a given volume v is uniformly bounded amongst all hyperbolic Dehn fillings on any given cusped hyperbolic 3-manifold.

(3) What is the largest volume $< v_\omega = 2.029883\ldots$ of a closed hyperbolic 3-manifold that does not arise from Dehn filling of $m004$ or $m003$? (This would allow us to make the result of Theorem 3.1 explicit.)

Experimental evidence suggests that the largest such volume is $2.028853\ldots$ for the closed manifold $m006(-5,2)$. (This is the largest such volume arising in the Hodgson-Weeks census of low volume closed hyperbolic 3-manifolds with shortest closed geodesic of length > 0.3, and also for Dehn fillings on the "magic manifold" s776.)

(4) (i) Are all hyperbolic Dehn fillings on $m004$ determined by their volumes, amongst Dehn fillings on $m004$?
(ii) Are all hyperbolic Dehn fillings on $m003$ determined by their volumes, amongst Dehn fillings on $m003$?
(iii) There are some equalities between volumes of Dehn fillings on $m003$ and $m004$, observed in [**27**], and shown in equation (5.6) above. The Meyerhoff manifold of volume $0.981368\ldots$ also arises as Dehn filling on both: $m004(5,1) = m003(-2,3) = m003(-1,3)$. Are these the only equalities between volumes of Dehn fillings on $m003$ and $m004$?

(These statements can be checked experimentally using SnapPy and Snap: for instance they are true for all hyperbolic Dehn fillings on $m003$ and $m004$ with volume < 2.0289.)

(5) For which cusped hyperbolic 3-manifolds N are there infinitely many hyperbolic Dehn fillings on N that are uniquely determined by their volumes amongst Dehn fillings on N?

Our methods give some further results on this question when the cusp shape τ lies in an imaginary quadratic number field. But for τ algebraic of higher degree, our approach would require new results on the distribution of gaps in the values of the quadratic form $Q(a,b) = |a + b\tau|^2$ for integers a, b. This question also arises in the subject of "Quantum Chaos", and it is conjectured that the successive gaps should be randomly distributed according to a Poisson distribution (see [**42**], [**41**], [**16**]). This would imply that arbitrarily large 2-sided gaps always exist.

(6) Are there sequences $x_i \to \infty$ with $N(x_i) = 1$ for all i?

(7) Are there sequences $x_i \to \infty$ such that the growth rate of the number $N_L(x_i)$ of link complements with volume x_i is faster than exponential, i.e with $\limsup_{i\to\infty} \dfrac{\log N_L(x_i)}{\log x_i} = +\infty$?

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